Autoresonant germ in dissipative system

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Abstract

We study an initial stage of autoresonant growth of a solution in a dissipative system. We construct an asymptotic formula of an autoresonant germ that is an attractor for autoresonant solutions. We present a moment of a fall and a maximum value of the amplitude for the germ. Numerical simulations are done.

1 Statement of the problem

In this paper we study an effect of a dissipation on a autoresonant solution. Let us consider the perturbed Duffing's oscillator with a dissipation term:

$$u'' + u + 4\varepsilon^{2/3}\beta u' - 2\sqrt{2}u^3 = 4\sqrt{2}\varepsilon f \cos(\omega t). \tag{1}$$

Here ε is a small positive parameter, $\omega = (1 - \varepsilon^{4/3}t)$, $\beta > 0$ and $f > 0$.

The orders of the dissipation and perturbation have special powers of ε. It does not lead to a loss of a generality because these terms contain additional parameters β and f. On the other hand it allows us to include these terms in the primary resonance equation [1].

The autoresonance means the essential growth of nonlinear oscillations due to a small oscillating force [2]. This phenomenon appears for (1) when the frequency of perturbation decreases slowly and $\beta = 0$ [2, 3, 4]. If $\beta > 0$ numerical simulations show that the growth presents on an initial stage. The direct analysis of the primary resonance equation allows one to estimate the

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maximum value of the solution. It was done for a slightly different equation in [5].

We study the solution of (1) in the form

$$u \sim \varepsilon^{1/3}\Psi(\tau)e^{i(t-\tau^2)} + \text{c. c.},$$

where $\tau = \varepsilon^{2/3}t$ and c.c. means a complex conjugate term. The amplitude Ψ of the oscillations is determined by the primary resonance equation

$$i\Psi' + (\tau - |\Psi|^2)\Psi + i\beta\Psi = f. \tag{2}$$

When $\beta = 0$ there exist increasing solutions of (2). These solutions are related to the autoresonance phenomenon, see [2, 3, 4].

To clear the problem we present results of numerical simulations when $\beta > 0$.

On the Figure 1 one can see the initial stage of growing for Ψ and the rapid fall. The growing part of the curve is related to the autoresonant phenomenon. The breakpoint shows the boundedness of the autoresonant growing in the system with dissipation.

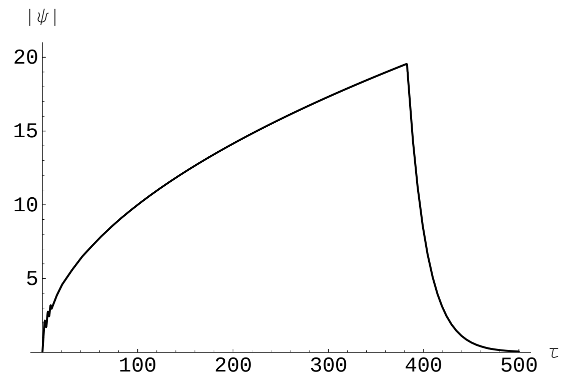

Figure 1: Modulus of solution for (2) with $\Psi|_{\tau=0} = 0,\ f = 1$ and $\beta = 0.05$.

Our goal is to obtain the autoresonant germ and study his behavior up to the moment of fall.

2 Result

If β is small then the autoresonant germ has the form

$$\Psi_G(\tau) \sim \left(\sqrt{\tau} + \frac{\sqrt{f^2-\beta^2\tau}}{2\tau}\right)\left(-\frac{\sqrt{f^2-\beta^2\tau}}{f} - i\beta\frac{\sqrt{\tau}}{f}\right) \times e^{\overline{\frac{i}{2\sqrt{\tau(f^2-\beta^2\tau)}}}}, \quad \tau \gg 1.$$

The life time of this germ is bounded by $\tau_* \sim \dfrac{f^2}{\beta^2}$ as $\beta \to 0$. This germ is an attractor for captured solutions. The captured solutions approach to the germ as $O\bigg(\exp\{-\beta\tau\}\bigg)$.

3 Asymptotic behavior of autoresonant germ

When $\beta = 0$ there exist pure algebraic solutions of (2) as $\tau \gg 1$. They were studied in [4]. Similar solutions of second order dissipativeless equations are called slowly varying equilibriums [6].

Let us construct a solution of (2) with a slowly varying leading-order term as $\beta > 0$. After the following substitution

$$\theta = \tau\beta^2, \quad \beta\Psi = \varphi$$

we obtain

$$i\beta^4\varphi' + (\theta - |\varphi|^2)\varphi + i\beta^3\varphi = \beta^3 f. \tag{3}$$

We construct the solution of the form

$$\varphi(\theta,\beta) = \Big(\sqrt{\theta} + \beta^3\rho_1(\theta) + \beta^4 R(\zeta,\theta,\beta)\Big)\exp\left\{i(\alpha_0(\theta) + \beta\alpha_1(\theta) + \beta^2 A(\zeta,\theta,\beta))\right\}, \tag{4}$$

where $0 < \beta \ll 1$ and $\zeta = \beta^{-3}\theta$ is a fast variable.

Substitute (4) into (3) and gather terms with the same order of β. It allows us to determine functions α_0, α_1 and ρ_1

$$\begin{aligned}
\sin(\alpha_0) &= -\frac{\sqrt{\theta}}{f}, \\
\alpha_1 &= \frac{1}{2\sqrt{\theta(f^2-\theta)}}, \\
\rho_1 &= \frac{\sqrt{f^2-\theta}}{2\theta}.
\end{aligned}$$

The residual terms R and A are solutions of the system

$$\begin{aligned}
R'_\zeta &= -\beta^2\rho'_1 - \beta\rho_1 - \beta^2 R + \beta^{-2}\Big(- f\sin(\alpha_0 + \beta\alpha_1 + \beta^2 A) + f\sin(\alpha_0) + \\
&\qquad \beta f\alpha_1\cos(\alpha_0)\Big), \qquad (5)\\
A'_\zeta &= -\beta\alpha'_0 - \beta^2\alpha'_1 - \beta^3\rho_1^2 - 2\beta\sqrt{\theta}R - \beta^5 R^2 - 2\beta^4\rho_1 R + \\
&\quad f\Big[\cos(\alpha_0 + \beta\alpha_1 + \beta^2 A) - \cos(\alpha_0)\Big]\left[\Big(\sqrt{\theta} + \beta^3\rho_1(\theta) + \beta^4 R\Big)^{-1} - \frac{1}{\sqrt{\theta}}\right].
\end{aligned}$$

The right hand side of equations contains slowly varying coefficients with respect to the fast independent variable ζ. Similar equations were studied

by applying WKB-method in [7]. The linearization of (5) gives the system with eigenvalues

$$\lambda_{1,2} = \pm i(2\theta)^{1/2}\sqrt[4]{f^2-\theta}\beta^{1/2} \mp i\frac{\sqrt[4]{\theta(f^2-\theta)}}{2\sqrt{2}(f^2-\theta)}\beta^{3/2} - \frac{1}{2}\beta^2 + O(\beta^{5/2}).$$

It shows the stability of the solution with respect to the linear approximation. It means that all captured solutions ψ are represented by

$$\psi \sim \left(\sqrt{\theta} + \beta^3\rho_1\right)\exp\{i(\alpha_0 + \beta\alpha_1)\} + O(\beta^4\exp\{-\beta^{-1}\theta\}), \quad \theta < f^2.$$

As a result we obtain the autoresonant germ

$$\Phi_G(\theta,\beta) \sim \left(\sqrt{\theta} + \beta^3\frac{\sqrt{f^2-\theta}}{2\theta}\right)\left(-\frac{\sqrt{f^2-\theta}}{f} - i\frac{\sqrt{\theta}}{f}\right)e^{\frac{i\beta}{2\sqrt{\theta(f^2-\theta)}}}. \quad (6)$$

The representation (6) allows us to estimate the life time of the germ

$$\tau_* \sim \left(\frac{f}{\beta}\right)^2.$$

The maximum amplitude of solutions that are captured into the autoresonance is

$$\max|\Psi| \sim \frac{|f|}{\beta}.$$

4 Numerical simulations

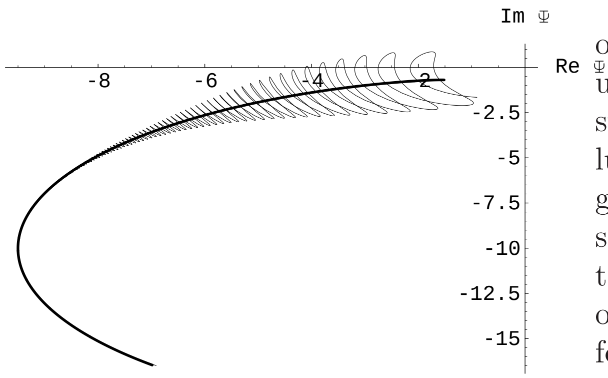

Figure 2: Sticking of the germ and numerical solution.

Here we present the result of numerical simulations. Figure 2 shows the exponential sticking of the numerical solution and the autoresonant germ. The heavy line corresponds to the germ and the thin line shows the behavior of numerical solution of (2) with for zero initial data, $f = 1$ and $\beta = 0.05$.

5 Control of autoresonance with dissipation

In this section we show a way to control the amplitude of the autoresonant oscillations in a system with the dissipation. In previous section we demonstrated that solutions fall. To prevent the fall one should stop the growth of the frequency of driving force.

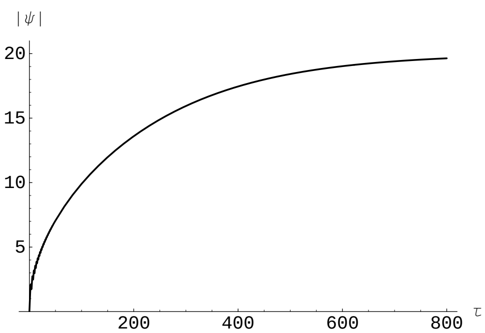

Figure 3: Control of autoresonant solution of (2) for $\Psi|_{\tau=0} = 0$, $f = 1$ and $\beta = 0.05$.

Here we present a result of numerical simulations. We suppose to stop the change of the frequency in equation (2). It leads to a capture of the amplitude. We demonstrate the numerical result when the term τ in (2) is changed by $(\beta/f)\tanh(f\tau/\beta)$ and

$$\omega = 1 - \varepsilon^{2/3}\frac{\beta}{2f\tau}\int_0^\tau \tanh\left(\frac{f\sigma}{\beta}\right) d\sigma.$$

6 Conclusions

In this paper we obtained the autoresonant germ in equation (1)

$$u \sim \varepsilon^{1/3}\Psi_G(\varepsilon^{2/3}t)\exp\{i(t - \varepsilon^{4/3}t^2)\}, \quad t \gg \varepsilon^{-2/3}.$$

We found that

$$\max |u| \sim \varepsilon^{1/3} \frac{f}{\beta}$$

and the solution is growing up to the moment $t_* \sim \varepsilon^{-2/3} f^2 \beta^{-2}$.

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